

MAS 117

Introduction to Algebra

Assignment 8

For handing in on 14 March 2007

*Write your name and student number at the top of your assignment before handing it in. Staple all the pages together. Post the assignment in the **RED** post-box on the **BASEMENT** of the Maths building before 16:00 on Wednesday.*

1 Each of the following (with the usual operations of addition and multiplication) fails to be a ring. In each case, you should prove this by showing one of the ring axioms which is not satisfied.

- $\mathbb{N}$, the set of natural numbers.
- $2\mathbb{Z} + 1$, the set of odd integers.
- The set of invertible 2×2 real matrices.
- The set of polynomials in which the coefficient of x^2 is zero.
- The set of vectors in real 3-dimensional space (the multiplication here is cross product).

2 Let R be a ring with more than one element.

- (a) Show that $a0 = 0$ for all $a \in R$.
- (b) Show that, if R has an identity element 1 for multiplication, then $1 \neq 0$.
- (c) Show that 0 cannot have an inverse for multiplication.

[This is why the multiplicative inverse law only requires that non-identity elements have inverses.]

3 Let R be a ring with identity. We say that two elements a and b of R are *associates*, written $a \sim b$, if $b = au$ for some unit u of R .

(a) Prove that $\sim$ is an equivalence relation.

(b) In the case that $R = \mathbb{Z}_{12}$, calculate the equivalence classes of $\sim$.

4 Prize question A cash prize is offered for the best solution received to this question. Do not hand this in with the rest of your coursework; give your solutions to me.

Find all positive integers m with the property that every unit a in $\mathbb{Z}_m$ satisfies $a^2 = 1$.

Axioms for a ring

Axioms for addition:

(A0) (*Closure law*) For any $a, b \in R$, we have $a + b \in R$.

(A1) (*Associative law*) For any $a, b, c \in R$, we have $(a + b) + c = a + (b + c)$.

(A2) (*Identity law*) There is an element $0 \in R$ with the property that $a + 0 = 0 + a = a$ for all $a \in R$. (The element 0 is called the *zero element* of R .)

(A3) (*Inverse law*) For any element $a \in R$, there is an element $b \in R$ satisfying $a + b = b + a = 0$. (We denote this element b by $-a$, and call it the *additive inverse* or *negative* of a .)

(A4) (*Commutative law*) For any $a, b \in R$, we have $a + b = b + a$.

Axioms for multiplication:

(M0) (*Closure law*) For any $a, b \in R$, we have $ab \in R$.

(M1) (*Associative law*) For any $a, b, c \in R$, we have $(ab)c = a(bc)$.

Mixed axiom:

(D) (*Distributive laws*) For any $a, b, c \in R$, we have $(a + b)c = ac + bc$ and $c(a + b) = ca + cb$.

A ring is *commutative* if it also satisfies

(M4) (*Commutative law*) For any $a, b \in R$, we have $ab = ba$.

A ring is a *ring with identity* if it also satisfies

(M2) (*Identity law*) There is an element $1 \in R$ with the property that $a + 0 = 0 + a = a$ for all $a \in R$. (The element 0 is called the *identity element* of R .)

A ring is a *field* if it is a commutative ring with identity and also satisfies

(M3) (*Inverse law*) For any element $a \in R$ with $a \neq 0$, there is an element $b \in R$ satisfying $ab = ba = 1$. (We denote this element b by a^{-1} , and call it the *multiplicative inverse* of a .)